Squaring the Circles: a Genealogy of *Principia* 's Dot Notation

Landon D. C. Elkind

Russell: The Journal of Bertrand Russell Studies, Volume 43, Number 1,
Summer 2023, pp. 42-65 (Article)

Published by Johns Hopkins University Press
DOI: https://doi.org/10.1353/rss.2023.a904086

SQUARING THE CIRCLES: A GENEALOGY OF *PRINCIPIA*'S DOT NOTATION

LANDON D. C. ELKIND
Political Science / Western Kentucky U.
Bowling Green, KY 42104, USA
LANDON.ELKIND@WKU.EDU

Russell derived many of his logical symbols from the pioneering notation of Giuseppe Peano. *Principia Mathematica* (1910–13) made these "Peanese" symbols (and others) famous. Here I focus on one of the more peculiar notational derivatives from Peano, namely, *Principia*'s dual use of a squared dot or dots for both conjunction and scope. As Dirk Schlimm has noted, Peano always had circular dots and only used them to symbolize scope distinctions. In contrast, *Principia* has squared dots and conventions such that some dots mark scope distinctions while others symbolize conjunction. How did this come to pass? In this paper I trace a genealogy of *Principia*'s square dots back to Russell's appropriation of Peano's use of circular dots. Russell never explicitly justifies appropriating Peano's notations to symbolize two distinct notions, but below I explain why Russell deployed Peano's dot notations in this manner. Further, I argue that it was Cambridge University Press who squared the circular dots.

Keywords: notation, *Principia Mathematica*, Bertrand Russell, Alfred North Whitehead, Giuseppe Peano, conjunction, scope, logic, connectives, Cambridge University Press

In this paper I discuss a puzzle regarding *Principia*'s notation for conjunction. *Principia*'s symbolism owes such a large debt to Giuseppe Peano's notations that we might call *Principia*'s notations "Peanese", though it should be recognized that *Principia*'s symbolism was influenced by other authors as well.[1] One glaring exception is *Principia*'s

[1] See SCHLIMM, "Peano on Symbolization, Design Principles for Notation, and the Dot Notation" (2021), for a discussion of Peano's views on symbols.

Russell: the Journal of Bertrand Russell Studies, n.s. 43 (summer 2023): 42–65 ISSN 0036-01631

E-ISSN 1913-8032

notation for conjunction, which is called "logical product" in that book. *Principia* uses neither Peano's notation of concatenation and parentheses for logical product, nor the symbol "**∧**", which is widely used today and is the dual of *Principia*'s notation for disjunction, "**∨**". *Principia* instead uses a square dot notation; what is more, this same notation is also used for scope. Why did they depart from their usual practice of following Peano, neglect to use a bolder version of the modern "∧" to mirror their notation for disjunction, and instead elect to do the aesthetically displeasing thing, to logicians at least, of using the same sign to symbolize different notions? On a more minor typographical note, in Peano's notations the dots are always circular, and Russell never used square dots in manuscripts or in print until the co-authored *Principia*. So why are the dots for logical product in *Principia* squared rather than circular?

In this paper I give a genealogy of *Principia*'s square dot notation. I trace the development of Russell's squaring of the circular dots of Peano and of his double use of the dot notation and argue that it admits of a perfectly rational explanation. First, I briefly explain *Principia*'s dot notation and its dual use for scope and logical product, and further show why their design choice is puzzling (Section I). Then I offer a historical explanation to solve this puzzle. Then, building on the work of Schlimm (see n. 1), I contextualize Russell's development by briefly reviewing Peano's notations for scope and for logical product (Section II). Then I trace the development of Russell's symbolism for logical product through his manuscripts and published writings prior to *Principia* (Section III). As we will see, Russell experimented with different symbols for logical product, including the "∧" that is widely used today.

With that background in place, I finally explain why Whitehead and Russell in *Principia* used dots for conjunction (Section IV). There are in fact two good reasons for this: first, using dots for scope and logical product can reduce, and never increases, the total number of symbols as compared with using the "∧"; second, using dots for arithmetic product was not uncommon in nineteenth-century mathematics books, and using dots for conjunction reflected that disciplinary convention for arithmetic in symbolic logic.

Finally, I discuss the puzzle of who squared the circular dots (Section V). The view defended here is that Cambridge University Press squared the circle dots, and did so to make the dots more readable and to help avoid misprints.

1. *PRINCIPIA*'S PUZZLING SQUARING OF PEANO'S CIRCULAR DOTS

Bertrand Russell derived many of his logical symbols from the pioneering notation of Giuseppe Peano. In the co-authored *Principia Mathematica*'s preface, among other places, this great debt is justly acknowledged:

> In the matter of notation, we have as far as possible followed Peano, supplementing his notation, when necessary, by that of Frege or by that of Schröder. A great deal of symbolism, however, has had to be new, not so much through dissatisfaction with the symbolism of others, as through the fact that we deal with ideas not previously symbolized. (*PM* 1: viii; *cf.* 4)

One idea with which Whitehead and Russell dealt with notationally was logical product, that is, conjunction of propositions (*PM*, *3). Peano had previously introduced symbolism for logical products; yet Whitehead and Russell did not use it. So Whitehead and Russell's general explanation for the introduction of new notations, that the ideas were not previously symbolized, does not apply in this case. Their new notation for logical product—the square dot—is thus puzzling.

Principia's new notation for logical product is even more puzzling when we consider that the same symbol has a dual use. *Principia* in fact uses a dot or dots to indicate conjunction, as in "**.**", "**:**", "**:.**", "**::**", and so on. However, *Principia* also uses these same dot notations to indicate scope distinctions generally. They indicate scope around binary truth-functional connectives and quantifiers, for example. So given the ill-formed string "$p \vee q \equiv r \supset s$", *Principia* has five ways of disambiguating scope with dots and making it a well-formed formula:

1. "$p \vee q \,.\, \equiv \,.\, r \supset s$" with "$\equiv$" as the main connective;
2. "$p \,.\, \vee \,.\, q \equiv r : \supset : s$" with "$\supset$" as the main connective and "$\vee$" as the secondary connective taking wider scope over "$\equiv$";
3. "$p \vee q \,.\, \equiv \,.\, r : \supset : s$" with "$\supset$" as the main connective and "$\equiv$" as the secondary connective taking wider scope over "$\vee$";
4. "$p : \vee : q \equiv r \,.\, \supset \,.\, s$" with "$\vee$" as the main connective and "$\supset$" as the secondary connective taking wider scope over "$\equiv$";
5. "$p : \vee : q \,.\, \equiv \,.\, r \supset s$" with "$\vee$" as the main connective and "$\equiv$" as the secondary connective taking wider scope over "$\supset$".

The general rule for *Principia*'s square dots is this: a greater number of dots around a connective (or making up the connective in the case of conjunction) always gives a connective wider scope over connectives with fewer dots. Ignoring the theorem-sign (which typically has the largest number of dots), the largest number of dots always occurs around the main connective.

Principia's dot notation should be distinguished from other "Peanese" notations that use dots. For example, Church has a convention for using dots on the right side of a connective to reduce the number of parentheses printed in representing a well-formed formula.[2] Curry offers a different dot notation where dots might occur on the left or right of connectives.[3] As Curry notes, his own "Peanese" symbolism for the formula

$$a \supset . \, b \supset : c \supset d \, . \supset e$$

has the consequence that, depending on whether or not the left dots take priority over groups of dots on the right of connectives, this formula might be

$$a \supset (b \supset ((c \supset d) \supset e)) \text{ or } (a \supset (b \supset (c \supset d))) \supset e.$$

Curry rightly points out that a simple convention can eliminate any apparent syntactic ambiguity in such formulas. Similarly, *Principia*'s notation has conventions to eliminate the possibility of syntactic ambiguity.[4]

On the other hand, the dots "." and ":" could in a different context symbolize conjunction.[5] In the formula

2 "A Set of Postulates for the Foundations of Logic" (1932), p. 354.

3 "On the Use of Dots as Brackets in Logical Expressions" (1937), pp. 26–7.

4 See *ibid.*, p. 26, where CURRY notes that *Principia*'s notation for dots does not have a convention for representing scope over an indefinite number of connectives; the implication $p_1 \supset p_2 \supset \ldots \supset p_n \supset p_m$ does not have a definite number of connective signs, so we do not know how many dots to place around the main connective. This is a practical inconvenience in metatheoretic investigations. But Whitehead and Russell were not directly undertaking such metatheoretical investigations in *Principia*. This is not a logical problem for *Principia* because their rules for notation supply the syntax for well-formed formulas. *Principia*'s notations for dots do not, and need not, give a grammar for metatheoretic symbolic descriptions of formulas any more than they give a grammar for metatheoretic descriptions in English.

5 Contra TURING ("The Use of Dots as Brackets in Church's System" [1942], p. 151), concatenation is never used for conjunction in *Principia*. For an alternative interpretation,

$$p : q . r$$

we have a conjunction of *p* with the conjunction of *q* and *r*, so that ":" is the main connective. In the formula

$$p . q : \equiv : q . p$$

there are scope-dots around the biconditional "≡" and conjunction-dots between the letters "*p*" and "*q*". Here the biconditional is the main connective because a greater number of dots surround it than the number of dots making up the conjunction-dot. So *Principia* has the same symbol, a square dot or dots, to symbolize both scope and conjunction: "Dots on the line of the symbols have two uses, one to bracket off propositions, the other to indicate the logical product of two propositions" (*PM* I: 9).

This raises the question of how to disambiguate formulas wherein equal numbers of scope-dots or conjunction-dots occur. *Principia* has conventions to determine the scope priority of dots when they occur in equal numbers either around a connective or making up a conjunction (*PM* I: 9–11). The convention is to classify dots into three groups in order of priority, (I) binary truth-functional connectives (other than conjunctions) and definitional equalities; (II) quantifier and description scope markers; (III) conjunctions. So if an equal number of dots occurs in each category, those of (I) have primary scope, (II) take secondary scope, and (III) have tertiary scope. Thus in the formula

$$(x) . \varphi x . \psi x . \supset . (x) . \varphi x$$

the dots around the conditional sign "⊃" belong to Group I and take priority over other equal numbers of dots from Groups II and III, since "⊃" is a truth-functional connective. The first quantifier dot belongs to Group II and has only the entire antecedent within its scope: it takes priority over equal numbers of conjunction-dots from Group III but is subordinate to equal numbers of connective dots from Group I. The conjunction-dot belongs to Group III and takes tertiary scope over "φx" and "ψx" in the antecedent. These conventions clarify what each occurrence of square dots symbolizes.[6]

which squares with Turing's reading, see LINSKY, "On the Use of Dots in *Principia Mathematica*" (2022).

6 Indeed, RUSSELL apparently settled on these conventions some years earlier; see, for example, "The Theory of Implication" (1906); **1** in *Papers* 5: 38, and "Mathematical Logic as Based on the Theory of Types" (1908); **22** in *Papers* 5: 609.

The conventions for dots can be summarized in a table organized by whether the dots in a given group have the first, second, or third greatest number of dots. Suppose that we have two connectives, C_1 and C_2, with equal numbers of dots. Then the possibilities for which gets wider scope is given in Table 1.

$C_2 \downarrow / C_1 \rightarrow$	**Group I**	**Group II**	**Group III**
Group I	tie	C_2	C_2
Group II	C_1	tie	C_2
Group III	C_1	C_1	tie

Table 1. Deciding which of two connectives with equal numbers of dots takes primary scope.

As we can see, the group numbers of connectives serve to determine which connective gets priority by breaking ties between connectives with equal numbers of dots. But where one connective is surrounded by more dots, that connective always takes wider scope. The decision procedure for determining which dots indicate wider scope is then as follows: *if a connective has a greater number of dots than another, that connective takes wider scope; otherwise, whichever one has the smallest group number takes widest scope*. Using a helpful convention from Linsky,[7] we can visualize this decision procedure by giving each dot a subscripted number that indicates its group. Then wherever we have equal numbers of dots, the one with the lowest group number takes priority. Applying Linsky's convention to the above formula gives us

$$(x) \cdot_{\mathrm{II}} \varphi x \cdot_{\mathrm{III}} \psi x \cdot_{\mathrm{I}} \supset \cdot_{\mathrm{I}} (x) \cdot_{\mathrm{II}} \varphi x,$$

which shows by visual inspection which dots take priority over the rest.

So Group III or conjunction-dots take priority only when they have the greatest number. Group I dots take priority when they are greater than or equal to the greatest number in other groups. Group II is trumped by a greater number of dots from other groups, or by an equal number of dots in a Group I occurrence (see Figure 1).

But why did Russell devise these grouping conventions for dots instead of just using "∧" for conjunction? He could also have just copied Peano and used concatenation to symbolize logical product. Then *Principia* would have needed just two groups for connectives instead of three. And

7 "The Notation in *Principia Mathematica*" (forthcoming).

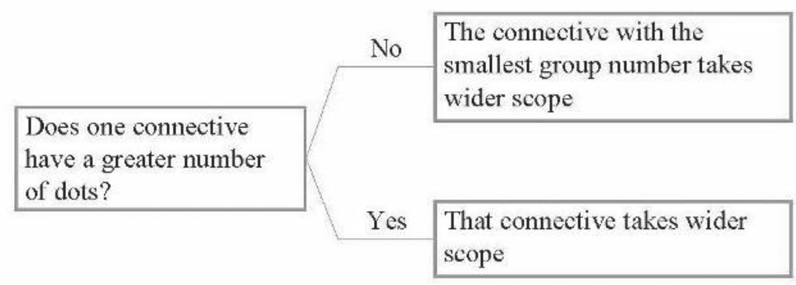


Fig. 1. Decision procedure for primary scope between two connectives.

why does it use the same square dot notation twice over for two distinct notions (scope distinctions and logical product)? And how do we know that it was Cambridge University Press that squared the circular dots?

Below I answer these questions. But first I review Peano's use of circular dots in Section II.

II. PEANO'S CIRCULAR DOT NOTATION

Peano of course developed much of his symbolism as parallel to exhaustive studies of earlier logicians and their notations: some volumes of the *Formulario* contain indexes of notations and references for their first use, and similarly for theorems.[8] This was part of his *Formulario* project of collecting all known mathematical theorems into his notation.[9]

Given Peano's rich and broad symbolic innovations, it is not surprising that *Principia* contained many Peano-inspired notations, or "Peanese" symbols, and accordingly promoted them even further. Of course, Peano's symbolisms were known to at least some mathematicians and logicians prior to 1910, including Couturat, Frege, Hilbert, Schröder, Russell, and Whitehead, and some of Peano's pupils like Burali-Forti, Padoa, and Pieri. Despite these authors' prior familiarity with Peano's symbols, *Principia* nonetheless somewhat distorted the interpretation of the same symbols.

One misinterpretation of particular interest to us is the prevailing confusion over Peano's use of dots. It has been and probably still is

8 See PEANO, *Formulario mathematico* (1895), 1: 127–44.
9 See KENNEDY, *Peano* (1980), pp. 44–5, for discussion and translations.

widely believed, incorrectly, that Peano used dots for conjunction.[10] This is not so, but it is easy to see how the misinterpretation might arise. Take Peano's formula

$$*24.\ a \supset b\ .\ b \supset c\ .\ c \supset d\ .\supset .\ a \supset d,$$ [11]

which is read aloud as "if *a* implies *b* and *b* implies *c* and *c* implies *d*, then *a* implies *d*" with "⊃" symbolizing implication. It is tempting to infer from how the formula is read aloud that the first and second dot symbols in the formula's antecedent signify conjunction, particularly since this formula would look identical if this were the case.

However, Peano never uses dots for conjunction—as Schlimm has argued, this would violate Peano's cardinal rule of symbolization "that each symbol must have a unique meaning"[12]—but typically uses concatenation for conjunction, though he sometimes uses "⌒". So "*ab*" stands for "*a* and *b*". This raises the question of how "*abc*" is to be interpreted. And this is where dots come in: they indicate which conjunction has primary scope. So "*a* . *bc*" and "*ab* . *c*" are both well-formed disambiguations of "*abc*". And of course the associativity of conjunction allows one to use "*abc*" conventionally, since the two disambiguations are equivalent. On the other hand, if we had "$a \supset bb \supset cc \supset da \supset d$", then the disambiguation matters. Theorem ∗24 above would be false if the first part of this string were disambiguated into "$a \supset bb\ .\supset .\ cc \supset da : \supset : d$", which would be "whenever it holds that if *a* implies *b* and *b*, then *c* and *c* implies *d* and *a*, it holds that *d*". And this is false, as *d* would not follow from that antecedent.

So, while it may look as though the first two dots in ∗24 symbolize conjunction, they really are scope markers. They disambiguate conjunctions, and conjunction is itself indicated by concatenations. This point is worth stressing because it shows that *Principia*'s appropriation of Peanese dots to symbolize conjunction and scope is not to be explained by suggesting that Russell just copied Peano's usage of dots. This would distort Peano's principles of symbolism on the one hand and oversimplify the history of Russell's evolving usage of Peanese dots on the other. A more accurate history that does justice both to Peano and Russell is desirable.

[10] An example is CAJORI, *A History of Mathematical Notations* (1952 [1st ed.,1929]), §689. See also the references in SCHLIMM, n. 40.

[11] *Formulario*, 1: §1.

[12] "Peano on Symbolization", pp. 118–19.

In Section III, I give a chronology of Russell's developing Peanese dot notations between his first encounter with that symbolism in 1900 and the publication of *Principia*'s first volume in 1910. Russell of course begins by adopting Peano's use of dots for scope, basically without adjustment, and later deploys it for conjunction as well. Russell never explicitly offers reasons for thus appropriating Peano's scope notations (among others), but there are at least two plausible and obvious rationalizations of which Russell would have known. I present these in Section IV.

III. A CHRONOLOGY OF RUSSELL'S DOT NOTATION

As we saw, Peano never used dots to indicate conjunction: he only used them to indicate scope. So when did Russell begin assigning square dots to double duty—for scope distinctions and for conjunction?

Russell's use of round dots for conjunction began quite early. In a 1901 French article, "Sur la logique des relations avec des applications à la théorie des séries" ([in English] **8** in *Papers* 3), Russell reserves concatenation for other notions. Russell is by this point familiar with Peano's notations, and with Schröder's symbolism, and so is definitely aware that concatenation has been used for logical products. For example, in the course of arguing in this article that the logic of Peano requires an explicit introduction of relations to be treated completely, Russell remarks that concatenation has only been used for logical multiplication by previous authors: "The juxtaposition of two letters has not hitherto possessed any meaning other than logical multiplication, which is not involved here" (in Peano's definition of function) (*Papers* 3: 314). Being aware of other authors' conventions regarding concatenation, Russell makes clear that he reserves concatenation for relational products: "It is necessary to distinguish $R_1 \frown R_2$, which signifies the logical product, from $\mathrm{R}_1\mathrm{R}_2$, which signifies the relative product" (*ibid.*, p. 316).[13]

On the other hand, we see Russell using round dots for conjunctions without explicitly saying so. For example, "Sur la logique des relations" has in the proof of ∗1·54 the line "(2) . (3) . (4) . ⊃ . Prop" (*ibid.*, p. 322), which Peano (following Russell's convention) typeset as "2 . 3 . 4 . Ɔ .

[13] In the same article (see *Papers* 3: 315) there occur concatenations like "ρu", involving some lower-case Greek letters standing for constants, and a lower-case letter to indicate that "u" is a class contained in the range of a relation "R". Here "ρ" is a constant that indicates the domain of a relation R, and "$\breve{\rho}$" stands for the range of R.

Prop". If concatenation represented conjunction in Russell's usage here, then these dots between "(2)", "(3)", and "(4)" would be superfluous.

Consider also the following formulas (*ibid.*, pp. 316–17):

$*2{\cdot}1.\ R_1, R_2\ \varepsilon\ \mathrm{Rel} \,.\supset:\, xR_1R_2z \,.=.\, \exists y \ni (xR_1y \,.\, yR_2z)$

$*2{\cdot}3.\ R_1\ \varepsilon\ \mathrm{Rel} \,.\supset:.\, R^2 = R \,.=:\, xRz \,.=_{x,z}.\, \exists y \ni (xRy \,.\, yRz)$

$*3{\cdot}5.\ x\varepsilon\breve{\varepsilon}y \,.=.\, \exists z \ni (x\varepsilon z \,.\, y\varepsilon z)$

$*3{\cdot}51.\ x\breve{\varepsilon}\varepsilon y \,.=.\, \exists z \ni (z\varepsilon x \,.\, z\varepsilon y).=.\, x, y\ \varepsilon\ \mathrm{Cls} \,.\, \exists xy$

In these formulas the dots that occur inside parentheses indicate conjunction. But these dots would be superfluous if dots only indicated scope distinctions for Russell. Assume for the sake of argument that Russell does indicate scope using dots and conjunction with concatenation in this piece. Then, since "*y*" does not stand for a relation, the string "xR_1yyR_2z" would be ill-formed unless it stood for the conjunction "xR_1y" and "yR_2z". There would be no need for a scope-dot between the two concatenated "*y*"s because, given the syntax of relation symbols and their terms, there is no other way to parse this string. So, if these dots really indicate scope, Russell inserted superfluous dots. Hence, these dots must indicate conjunction rather than marking scope. Likewise for dots that occur within parentheses in the other formulas.

This is entirely unlike the typical example formula ∗24 from Peano discussed above: there it would be well-formed to give one or more conditionals wider scope over conjunctions, even though the formula would then be false. In contrast, there is no disambiguation to make of scope in these cases: only one disambiguation is possible, or else the string within parentheses is ill-formed. So Russell must be using these dots to symbolize conjunction rather than to mark scope distinctions.

In "Sur la logique des relations" we find uses of two stacked dots (":") to indicate conjunction, as in ∗3·7·81·82, and even one use of three dots (":.") for it, in ∗3·8 (*Papers* 3: 317–18). So in that respect Russell's early usage matches that of *Principia*; compare, for instance, the proofs of ∗4·4·43·44. On the other hand, in this essay and in others from this period, we only see rounded Peanese dots, and, as in Figure 2 below, Russell's manuscripts from this period do not show signs of squaring or even thickening of dot notations.

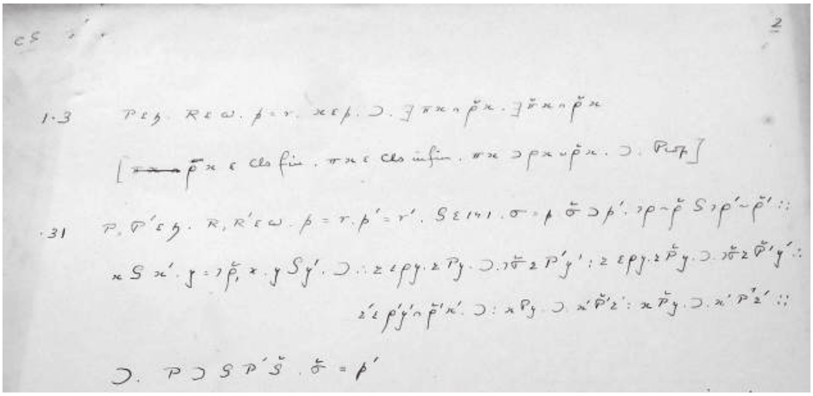


Fig. 2. Dot notation in Russell's "Continuous Series" manuscript (1901–02), RA1 230.030820.

As Gregory H. Moore pointed out in his editorial headnote to "On Likeness" (**15** in *Papers* 3: 439), in 1902 Russell used "∧" for class intersection (not for conjunction as it is widely used today). Specifically, in this manuscript (*ibid.*, p. 440) Russell uses a thicker wedge, "**∧**", as in the definition

$$*1\cdot5.\ R\ \varepsilon\ \mathrm{Rel}\ .\supset.\ \mathrm{Rel}‘R = \mathrm{Rel} \wedge P \ni (P \supset R)$$

Here the thick wedge indicates the intersection of the class of relations Rel with the class of relations *P* that contain *R*, including *R* itself.

To symbolically separate relation and class intersection, Russell also uses the different symbol, "∩", for relation intersection in "On Likeness" manuscript. And in the manuscript itself, a thickened wedge occurs in ∗1·3 (see Figure 3 below). This would make thickening not just a device of Russell's printer, but of Russell himself. Curiously, Russell still uses Peanese dots for conjunction (as in ∗2·57 [*Papers* 3: 441]) despite using "⌒" for disjunction.

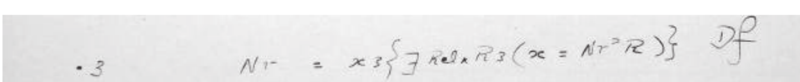


Fig. 3. Wedge notation in Russell's "On Likeness" manuscript (1902), RA1 230.030830.

In the 1903 "Classes" manuscript (**1** in *Papers* 4), Russell switches all of these uses except for the dot notation, which is still used for conjunction and scope: Russell here uses "∨" for disjunction and "◡" for class and relation union. Dots are still used for conjunction and scope, and this 1903 usage continues through *Principia* and beyond.

On the other hand, as has been noted by Elkind and Zach,[14] in this same 1903 manuscript Russell introduces "**∧**‘*m*" for a class-operator—the conjunction of a class *m* of propositions (∗14·8)—and similarly introduces "**∨**‘*m*" for its dual class-operator—for the disjunction of a class *m* of propositions (∗14·81). Russell was thus a hair's breadth away from having the idea of using a thickened "∧" for conjunction as the dual of a thickened "∨" for disjunction. Why did Russell not seize this aesthetically pleasing opportunity?

Russell's indifference to using "∧" is partly explained by the fact that the notation "rarely occurs" ("Classes"; *Papers* 4: 20). Russell also seems not to have used "∧" for anything like conjunction anywhere else. It largely drops out of the picture after these pre-*Principles* manuscripts. The "∧" symbol was thus not much on Russell's mind from 1902 through the publication of *Principia* in 1910, so far as the textual record indicates, after composing this one manuscript where it occurs. Furthermore, there are at least two reasons for Russell to use dots for conjunction, as we will see in Section IV.[15]

IV. WHY RUSSELL USED DOTS FOR CONJUNCTION

In his published writings and surviving manuscripts, Russell never offers a rationale for adopting the dual use of dots for conjunction and scope. However, there are two reasons that may have influenced Russell's decision to use dots in this dual way, and it is hard to believe that Russell was unaware of them. So it might appear to us today, given the widely-used and visually-dual notations "∧" and "∨", that Russell missed an opportunity to reinforce with his notation, by using visually dual symbols, the logical duality of conjunction with disjunction. But his approach admits of two understandable rationales. The first rationale is practical; the other is conventional.

14 "The Genealogy of '**∨**'" (2022), §4.

15 One might think that Russell eschewed using "∧" for conjunction to symbolically distinguish the symbols for conjunction and the empty class, "Λ". This cannot be right because Russell has "**∨**" and "**V**" as symbols for disjunction and the universal class in *Principia*.

The practical reason for using dots in Russell's dual way is that the dual usage of dots, as compared with using "∧" and "∨", sometimes reduces, and never increases, the total number of dots in a given formula. Consider the following formula, rendered using modern notation for scope and conjunction:

$$((p \wedge q) \wedge (r \wedge s)) \supset ((p \wedge s) \wedge (r \wedge q)).$$

In *Principia*'s notation this formula becomes

$$p \,.\, q : r \,.\, s : \supset : p \,.\, s : r \,.\, q,$$

where the dots around the implication sign indicate scope and all the other dots indicate conjunctions. Note that these dots take precedence over conjunction-dots where the number of dots is equal. Now if we use the wedge for conjunction, we will then need the relevant scope markers indicating the scope to be given to each occurrence of "∧". The formula would then be

$$p \wedge q \,.\, \wedge \,.\, r \wedge s : \supset : p \wedge s \,.\, \wedge \,.\, r \wedge q,$$

where all dots indicate scope. Notice that the formula is wider than its counterpart that uses dots alone and nonetheless requires the same number of scope-dots around the main connective. So formulas using wedges require more horizontal space than their wedge-free counterparts (assuming all else, like the spacing between symbols, is equal).[16]

Of course, if one adopted a convention between the connectives themselves, so that "⊃" takes wider scope over "∧" where they have equal numbers of dots, then the above formula becomes

$$p \wedge q \,.\, \wedge \,.\, r \wedge s \,.\, \supset \,.\, p \wedge s \,.\, \wedge \,.\, r \wedge q.$$

Note that switching from conjunction-dots to conjunctive wedges does not always decrease the number of scope-dots around the main connective. Consider another formula rendered in modern notation,

$$(p \wedge q) \supset ((p \vee q) \vee (p \vee q)).$$

In *Principia*'s notation this formula becomes

[16] It is worth noting that Peano's notation reduces the number of dots as compared with the conventions in *Principia*. The formula given above, for example, becomes $pq \,.\, rs : \supset : ps \,.\, rq$, which has fewer dots around the main connective and takes less horizontal space. Thanks to Dirk Schlimm for pointing this out.

$$p \,.\, q :\supset: p \vee q \,.\vee.\, p \vee q,$$

while in wedge-liberated notations it becomes

$$p \wedge q :\supset: p \vee q \,.\vee.\, p \vee q,$$

where the main connective is ⊃. However, we do have the following:

Proposition.[17] *Suppose we have two propositional languages* $L_.$ *and* $L_\wedge$ *and syntactic rules for well-formed formulas that are identical except where conjunctions occur. Suppose further that the conventions for dots in* Principia *are followed (*PM *1: 9–11). Then the main connective of any formula* $\varphi_\wedge$ *in the language* $L_\wedge$ *will have at least as many scope-dots around it as its analogue* $\varphi_.$ *in the language* $L_.$ *has.*

Proof. Suppose we replace a conjunction-dot in $\varphi_.$ with a wedge. Call the resulting formula $\varphi_\wedge$. We show by induction on the length of the formula that $\varphi_\wedge$ has a greater or equal number of scope-dots than its analogue $\varphi_.$ does. In the base case where "." is the only connective, no extra dots are needed: we go from $p \,.\, q$ to $p \wedge q$. Now assume the proposition holds for any formula of shorter length than $\varphi_.$. There are three cases:

1. $\varphi_.$ is $B \,._n\, C$ where $._n$ is n-many dots (so $._4$ = ::) in this case, either (a) we replace $\varphi_.$ with $\varphi_\wedge = B \,._n \wedge ._n\, C$ and the result holds, or (b) we replace some $._n$ inside either B or C and the result holds by the inductive hypothesis.
2. $\varphi_.$ is $B \,._n \vee ._n\, C$: in this case, only a conjunction in B or in C will be replaced and the result holds by the inductive hypothesis.
3. $\varphi_.$ is $\sim B$: in this case, only a conjunction in B will be replaced and the result holds by the inductive hypothesis.

We may assume that $._n$ (or $\wedge$ in $L_\wedge$), $\vee$, and ~ are the primitive propositional connectives in $L_.$ and $L_\wedge$. So we are done.

Note that one might adopt wedge-oriented conventions that would decrease the number of dots. For example, we might adopt the convention that wherever equal numbers of dots occur around "⊃" and any

[17] The explanation for this proposition is this: "Because we always need two parentheses to enclose a subformula, but only one group of dots to separate a subformula, the dot notation uses fewer symbols" (SCHLIMM, p. 116).

other propositional connective, the implication sign has wider scope. Then (see p. 54 above)

$$(p \wedge q) \supset ((p \vee q) \vee (p \vee q))$$

becomes

$$p \wedge q \supset (p \vee q) \vee (p \vee q)$$

and its analogue using *Principia*-style dots for scope is

$$p \,.\, q \,.\supset .\, p \vee q \,.\vee .\, p \vee q.$$

This does not undercut the reasoning that establishes our proposition above. The analogue in *Principia* of

$$p \wedge q \supset (p \vee q) \vee (p \vee q),$$

on that additional convention, is

$$p \wedge q \,.\supset .\, p \vee q \,.\vee .\, p \vee q,$$

and, in general, replacing a wedge with a conjunction-dot never raises the number of dots around the main truth-functional connective. And similar reasoning establishes the same result for formulas of quantifier logic.

I conjecture that Russell knew this proposition obtained and realized that replacing conjunctive wedges with conjunction-dots could decrease the number of symbols and would never increase it. Accordingly, there was little reason based in notation, besides possibly aesthetic ones, for him to prefer wedges over conjunction-dots. Indeed, it would be harder, mainly for the typesetter, to exploit the number of wedges occurring in a group as the number of dots occurring in a group. With modern technology, it is simple to typeset wedges in groups, as in

$$\wedge \;\; \wedge \;\; \wedge\wedge \;\; \wedge\wedge$$
$$\wedge, \wedge, \wedge\wedge, \wedge\wedge, \wedge\wedge\wedge,$$

but these would be more nightmarish to print using metal type than groups of dots.[18] Now direct inspection of *Principia* shows that Cambridge University Press was capable of complex printing jobs. Still, horizontal space was at a premium in Russell's writings leading up to and including *Principia*. Many of Russell's logical works included longer

[18] I thank a reviewer for inviting me to address this point.

formulas up to the width of a page in need of typesetting. Given the printers' convenience and need, it is understandable that Russell was less concerned with typographical elegance than shrinking the number of symbols to be typeset horizontally.

Again, Russell never states Proposition 4.1 explicitly. But Russell was very likely aware of the fact that his appropriation of Peanese dot notation for conjunction and scope reduced the number of symbols, although, again, not as compared with Peano's symbols (see n. 16). However, if one was insistent on not using concatenation for conjunction, as the authors of *Principia* were, then Peano's notation would not be available as an alternative.[19]

The second reason that Russell likely used dots for conjunction, as well as using them for scope, was the causal influence of conventions for using dots for products in mathematics and in logic. In the discipline of mathematics, although most notations historically have not enjoyed universal adoption, there was a longstanding tradition of using "·" or "." for arithmetic products. This practice dates back at least to Leibniz.[20] In the late nineteenth and early twentieth centuries, it was a common practice in a variety of areas, at least in texts first published in English.[21]

Thanks to Griffin and Lewis,[22] it is also known that Russell was familiar with enough of the mathematics of his day to be aware of the disciplinary convention. Admittedly, Russell did not preserve many memorabilia from his formal education in mathematics at Cambridge when,

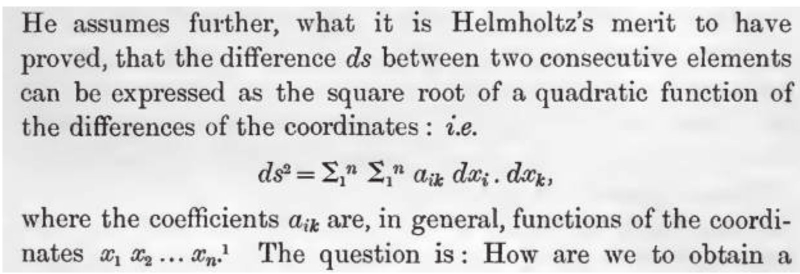
19 Indeed, *Principia* rarely concatenates symbols. One exception is the notation for relations in extension, as in "xRy" in *21·42. Also, there are sometimes concatenations of circumflexed symbols, as in *21·01, where we find "$f\{\hat{x}\hat{y}\psi(x, y)\}$", and in *21·08.

20 See CAJORI, §546.

21 See TODHUNTER, *Spherical Trigonometry* (1871), p. 16; CASEY, *A Treatise on the Analytical Geometry of the Point, Line, Circle, and Conic Sections* (1885), p. 69; IBBETSON, *An Elementary Treatise on the Mathematical Theory of Perfectly Elastic Solids* (1887), pp. 22–3; LACHLAN, *An Elementary Treatise on Modern Pure Geometry* (1893), pp. 12–13; HARKNESS AND MORLEY, *A Treatise on the Theory of Functions* (1893), pp. 3–4; and WHITTAKER, *A Course of Modern Analysis* (1902), p. 32. Here I mention only works originally published in English. This is to ensure that different printing conventions between countries, or an editor or translator's decision, are not influencing the notations in the cited works. Accordingly, my claim only concerns whether this dot convention is an established one in English-language works. Note that some works originally published in another language and translated into English follow similar dot conventions; e.g., HARNACK, *An Introduction to the Study of the Elements of Differential and Integral Calculus* (1891), pp: 48–9.

22 "Bertrand Russell's Mathematical Education" (1990), pp. 51–2.

for instance, he was studying for the Tripos examination.[23] Griffin and Lewis also note that Russell was quite familiar with some texts wherein these conventions were followed. For example, he read Charlotte Angas Scott's *An Introductory Account of Certain Modern Ideas and Methods in Plane Analytical Geometry* (1894). He reviewed A. E. H. Love's *Theoretical Mechanics* in *Mind* (1897). And he read James Harkness and Frank Morley's *An Introduction to the Theory of Analytic Functions* (1898).[24] All three works use dots for arithmetic products.[25] Russell was familiar enough with the convention to use dots for arithmetic product, and not for scope or conjunction, in *An Essay on the Foundations of Geometry*, a revised version of his Fellowship dissertation (see Figure 4).


He assumes further, what it is Helmholtz's merit to have proved, that the difference ds between two consecutive elements can be expressed as the square root of a quadratic function of the differences of the coordinates: *i.e.*

$$ds^2 = \Sigma_1^n \, \Sigma_1^n \, a_{ik} \, dx_i \,.\, dx_k,$$

where the coefficients a_{ik} are, in general, functions of the coordinates $x_1 \, x_2 \ldots x_n$.[1] The question is: How are we to obtain a

Fig. 4. Dot notation in *An Essay on the Foundations of Geometry*, §22 (1897).

In this formula Russell uses both concatenation and dots to represent multiplication. The dot here also serves as a scope marker to distinguish

$$\sum_1^n \sum_1^n a_{ik} dx_i \,.\, dx_k = \sum_1^n \Big(\sum_1^n a_{ik} dx_i\Big) dx_k$$

from

$$\sum_1^n \sum_1^n a_{ik} (dx_i \; dx_k).$$[26]

[23] *Ibid.*, p. 55.

[24] On Russell's familiarity with these works, see *ibid.*, pp. 59, 62 and 65, respectively.

[25] See SCOTT, pp. 16–17; LOVE, pp. 36, 79, 87, 124–5; and HARKNESS AND MORLEY, *An Introduction*, pp. 40–1.

[26] Thanks to an anonymous reviewer for suggesting that I expand on this point.

This shows Russell was familiar with concatenation and dots as disciplinary conventions in mathematics for representing products. But sometimes (in *EFG*, §22, for example), Russell uses a dot for multiplication where scope ambiguity is already eliminated, as in

$$\frac{n \cdot (n - 1)}{2},$$

and we find just concatenation used for representing product, as in

$$ds^2 = \sum dx^2 (1 + \frac{a}{4} \sum x^2)^2$$

Those working on logic in the nineteenth century occasionally used dot notations, as Cajori noted in 1929.[27] DeMorgan, Peirce, and Schröder, for example, sometimes use dots for scope in accordance with the longstanding practice of using dots to mark scope in arithmetic products. We saw that Peano never used dots for conjunction. But Schröder occasionally used dots for conjunction, as in Theorem 21, $\vdash a \cdot 1 = a$, and in Theorem 22, $\vdash a \cdot 0 = 0$.[28] Similarly, although MacColl uses "×" for conjunction, he introduces "·" as a synonym for it.[29] Additionally, MacColl uses ":" for implication. Russell was certainly very familiar with Schröder and MacColl, as well as with the disciplinary convention in logic of using dots for logical products (in some cases) much as they were commonly used for arithmetical products in mathematical contexts. Given the close analogy of logical and arithmetic product, the conventions of authors in both disciplines, and Russell's intimate familiarity with both, it is unsurprising that Russell deployed dots for conjunction.

Besides being unsurprising, Russell's use of dots (or some symbol) for conjunction was necessitated by his use of concatenation for other logical notions. As we saw above, Russell used concatenation for relations, relational products, membership, and a whole host of other logical notions (often relying on capitalization or constants to distinguish various notions). Russell's symbolism would be practically unreadable if concatenation were also used for conjunction, especially given how

27 *A History of Mathematical Notations*, §§677, 681, 685–6.

28 *Volesungen über die Algebra der Logik* (1891), §29. I thank a reviewer for pointing out these theorems and for impressing upon me the importance of conventions in the logic discipline as an additional influence on Russell's use of dots for conjunction.

29 "Symbolical Reasoning" (1880), p. 49.

frequently conjunction occurs in his logical works like *Principia*. So Russell needed some alternative to the convention used by the majority of logicians in his day, and he naturally enough chose one that combined different conventions of such notable logicians of his day as Peano, Schröder, and MacColl (albeit infrequently). Russell's hybrid convention was in good (or at least traditional) mathematical taste to boot.

The textual evidence supports that Russell's use of dots for conjunction was inspired by the disciplinary conventions of mathematicians and (some) logicians of using dots for (arithmetic or logical) products. In "The Theory of Implication" (1906), Russell calls the dots for conjunction the symbol for a "propositional product", and similarly calls "∨" for disjunction the symbol for a "propositional sum", in a section titled "Multiplication and Addition" (see *Papers* 5: 37–44). In Section VI of "Mathematical Logic as Based on the Theory of Types" (1908), Russell similarly calls the dots for conjunction "logical product" and "∨" for disjunction "logical sum" (*ibid.*, pp. 608–12). *Principia* likewise calls conjunction-dots the sign for "logical product" and "∨" the sign for "logical sum" (*PM* 1: 97, 114). The terminology was clearly not accidental, but was chosen to relate the logical symbols to common symbols for analogous arithmetic ideas.

One might wonder why Russell did not use "+" for logical sum if I am right that "×" was chosen partly to reflect the disciplinary convention regarding arithmetic notions. But *Principia* uses "+" and "×" for arithmetic notions. This precludes using "+" for logical sum, and it took some notational exploration for Russell to finally settle on "∨" for disjunction.[30] Perhaps this was in part because there was not, so far as I know, a disciplinary convention of using a different symbol for arithmetic addition.

The choice to use dots for conjunction was then independent of how to symbolize disjunction or propositional addition, even though in both cases Russell was keen to stress informally, and symbolically when he could, that these are the logical notions such that the symbols for the analogous arithmetic ideas would be more familiar to most mathematicians. Additionally, since dots were already being used for scope, it made good sense to use them again for conjunction, although it would also have made sense to use "∧" given the use of "∨", Either of those choices would have made more sense than using the symbol "×" which

[30] See ELKIND AND ZACH, §IV.

belonged to the later, specialized parts of *Principia* and not the maximally general logical sections.

To summarize, Russell's dual use of Peanese dot notation can be explained as follows: Russell took the dot notation for (arithmetic and logical) product then conventional among mathematicians and (some) logicians, appropriated it for conjunction, and wed this notation to Peanese scope-dots, which he had already adopted in an effort to save space and avoid overloading concatenation with too many logical notions for his symbolism to be readable.

V. WHO SQUARED THE CIRCLES?

There is one further question about Russell's dot notation. Why are the dots squared in *Principia*? Peano always used circular dots.

Did Whitehead and Russell square the circle dots? This cannot be ruled out definitively because, unfortunately, not much of the *Principia* draft material survives. Whitehead requested in his will that all his papers be destroyed after he died, and Whitehead's wife, Evelyn, complied. And Russell habitually discarded most of the manuscript material for *Principia* that he sent to Cambridge University Press.[31] However, in the few surviving *Principia* manuscripts[32] and in Whitehead's and Russell's other manuscripts and publications prior to *Principia*, there is no indication of circular dots evolving into a squared shape. Given there is no surviving evidence of squaring dots by Whitehead or Russell, the fact that there are square dots in *Principia* is, I conjecture, due to an in-house stylistic decision by Cambridge University Press.

There are of course plenty of circular dots in *Principia*. These occur in the text's English remarks and in some of its logical notations. There were printing issues related to these circular dots: Cambridge University Press had to insert some circular dots by hand—in all 750 copies of *Principia*'s first volume! These were needed over capitalized lambdas

[31] LINSKY AND BLACKWELL, "Russell's Corrected Page Proofs of *Principia Mathematica*" (2019), p. 141.

[32] I.e., in the three half-leaves reproduced in LINKSY AND BLACKWELL, "New Manuscript Leaves and the Printing of the First Edition of *Principia*" (2005), pp. 143–5, 148, and a sole leaf reproduced in GRIFFIN, LINSKY AND BLACKWELL, eds. "Illustrations. Manuscripts Relating to *Principia Mathematica*" (2011), p. 81.

to make the symbol "$\dot{\Lambda}$" representing an empty relation in extension.[33] Some examples of this are visible in *Principia*'s first edition, where one can see that dots over capitalized lambdas are off-centre and sometimes occur more to the left, whereas elsewhere they occur more to the right. The brittleness of printing type caused some smaller and circular dots to be omitted,[34] as indicated in the Errata to Volume I: "p. 218, last line but for one, *for* 'Λ' *read* '$\dot{\Lambda}$' [owing to the brittleness of the type, the same error is liable to occur elsewhere]." Given that there is no similar erratum for dots over lambdas in *Principia*'s second and third volumes, Linksy and Blackwell infer that Cambridge University Press probably ordered stronger materials from the type foundry.[35] The printers doubtless thanked them.

This explains what happened in Volume II and III with overset circular dots. But there are square dots in Volume I that, unlike some overset circular dots, were reliably printed. How did this come about? When Cambridge University Press first attempted to print *Principia*'s dot notations for conjunction and for scope, they likely anticipated (or discovered by error very early on) that new, thicker metal type was needed for the job so as to avoid the printing errors similar to those encountered with overset circular dots. Indeed, some material produced by Cambridge University Press has square dots, like the recently discovered page proofs for Volumes I and II.[36] Cambridge University Press likely anticipated (or discovered quickly) that the brittleness of the usual metal type for dots would make printing many hundreds of dots difficult. This printing job would be far easier with thicker metal type as opposed to ordinary type used to print periods at the end of sentences.

Also, Whitehead and Russell likely told Cambridge University Press to make their dot notation for scope, which is ubiquitous in all three volumes of *Principia*, clearly visible. Russell once described himself as "fussy" about such matters.[37] And we know that Russell sometimes gave explicit instruction to editors and printers about how notations should appear. For example, leading up to the publication of "The Theory of Implication" in the *American Journal of Mathematics*, Russell wrote to

33 LINSKY AND BLACKWELL, "Russell's Corrected Page Proofs", pp. 152–3.
34 Compare, say, *25·103 and *25·105.
35 "Russell's Corrected Page Proofs", p. 154.
36 *Ibid.*, p. 153.
37 For a fuller discussion of Russell's concern about "accidentals" like spacing and punctuation, see BLACKWELL, "'Perhaps You Will Think Me Fussy . . .'" (1983), §5.

its editor, Frank Morley, concerning the symbol "∨" for disjunction: "It is very desirable that it ['∨'] should be pointed, not round; but otherwise it doesn't matter much whether it is large or small, though I think that it would be better small" (3 Sept. 1905; *Papers* 5: 15). In that same letter to Morley, Russell said this about the dot notation: "As for the dots, they may be arranged in any shape: only their *number* is important. But they ought *not* to be all on the line, thus: '. . .'" (*ibid.*). Russell may have been fussy about the dots not being on one line, but did not seem to care what shape each dot was or how the dots were arranged on the page (except that the dots must not occur in a single line). Similar instructions were probably given to Cambridge University Press. But Russell did not suddenly start caring about whether the dots were circles or squares between this 1905 letter and the 1910 publication of *Principia*'s first volume.

So, to solve problems associated with printing *Principia*'s dot notations, and to satisfy the authors' need for the dots to be clearly visible and not all on one line, Cambridge University Press likely ordered thicker metal type (or used thicker metal type that they had on hand) for dots in *Principia*. The dots produced with this thicker metal type just happened to be square. Hence, thanks to Cambridge University Press, *Principia* has the thickened (by design) and square (by accident) dots.

The difficulties associated with printing *Principia* also explain one further fact about its dot notation. As we saw, *Principia* uses the dot notation for two distinct logical notions. This might seem to be bad logical style: different notations should be used for different ideas. Why did they not use thickened circles for one notion and thickened squares for another?[38]

Practical consideration of the printers' needs perhaps overrode this norm of good logical style. It may be that thickened square dots and thickened circle dots would be hard to distinguish, especially given how ubiquitous dots are in *Principia*. Perhaps more importantly, it would have entailed extra expense to order additional thickened metal type; it would also have been a substantial inconvenience to the printers to set both thickened square and thickened circle dots tens of times in the same page. No doubt Cambridge University Press could have easily prevailed on the authors to adopt a convention for dots that would prevent

[38] I thank a reviewer for inviting me to address the possibility that differently shaped dots might have been used for the distinct notions of conjunction and scope.

compounding the expense and trouble demanded by the already difficult job of printing *Principia*—a job for which, we should recall, Whitehead and Russell were already paying a considerable sum![39]

It bears mentioning again that we can only conjecture as to who squared *Principia*'s dot notations because there are few surviving materials from the production of *Principia*. From what does survive, I conclude that it was Cambridge University Press who, by accident, squared the circles.

Author's note: This research was undertaken when I was an Izaak Walton Killam Postdoctoral Fellow in Philosophy at the University of Alberta. Thanks to Katalin Bimbo, Bernard Linsky, Dirk Schlimm, and two anonymous reviewers for their helpful comments.

WORKS CITED

BLACKWELL, KENNETH. "'Perhaps You Will Think Me Fussy . . .': Three Myths in Editing Russell's *Collected Papers*". In H. J. Jackson, ed. *Editing Polymaths: Erasmus to Russell*. Toronto: Committee for Conference on Editorial Problems, 1983.

CAJORI, FLORIAN. *A History of Mathematical Notations: Notations Mainly in Higher Mathematics*. Vol. 2. 3rd imp. Chicago: Open Court, 1952. 1st ed., 1929.

CASEY, JOHN. *A Treatise on the Analytical Geometry of the Point, Line, Circle, and Conic Sections, Containing an Account of Its Most Recent Extensions, with Numerous Examples*. Dublin: Hodges, Figgis; London: Longmans, Green, 1885.

CHURCH, ALONZO. "A Set of Postulates for the Foundation of Logic". *Annals of Mathematics* 33 (1932): 346–66.

CURRY, H. B. "On the Use of Dots as Brackets in Logical Expressions". *The Journal of Symbolic Logic* 2 (1937): 26–8.

ELKIND, LANDON D. C. "The Cost of Publishing *Principia Mathematica*". *Daily Nous*, 6 July 2016; https://dailynous.com/2016/07/06/costs-publishing-principia-mathematica-landon-elkind/.

—, AND RICHARD ZACH. "The Genealogy of '∨'". *The Review of Symbolic Logic* [FirstView] (2022): 1–38; doi:10.1017/S1755020321000587.

GRIFFIN, NICHOLAS, AND ALBERT LEWIS. "Bertrand Russell's Mathematical Education". *Notes and Records of the Royal Society of London* 44 (1990): 51–71.

—, BERNARD LINSKY, AND KENNETH BLACKWELL, eds. "Illustrations: Manuscripts Relating to *Principia Mathematica*". *Russell* 31 ["*Principia Mathematica* at 100" special issue] (2011): 81–4.

HARKNESS, JAMES, AND FRANK MORLEY. *A Treatise on the Theory of Functions*. New York and London: Macmillan, 1893.

—, *An Introduction to the Theory of Analytic Functions*. London and New York: Macmillan, 1898.

HARNACK, AXEL. *An Introduction to the Study of the Elements of Differential and Integral Calculus*. London: Williams and Norgate, 1891. 1st German ed., 1881.

IBBETSON, WILLIAM JOHN. *An Elementary Treatise on the Mathematical Theory of Perfectly Elastic Solids, with a Short Account*

[39] See ELKIND, "The Cost of Publishing *Principia Mathematica*" (2016) for a calculation in 2016 dollars of what the authors paid to publish *Principia*.

of Viscous Fluids. London and New York: Macmillan, 1887.

KENNEDY, HUBERT C. *Peano: Life and Works of Giuseppe Peano*. Dordrecht, Holland: D. Reidel, 1980.

LACHLAN, ROBERT. *An Elementary Treatise on Modern Pure Geometry*. London and New York: Macmillan, 1893.

LINSKY, BERNARD. "The Notation in *Principia Mathematica*". In Edward, Zalta and Uri Nodelman eds. *Stanford Encyclopedia of Philosophy Archive*, Fall 2022 ed.; https://plato.stanford.edu/archives/fall2022/entries/pm-notation/

—. "On the Use of Dots in *Principia Mathematica*", forthcoming [in *Russell*].

—, AND KENNETH BLACKWELL. "New Manuscript Leaves and the Printing of the First Edition of *Principia Mathematica*". *Russell* 25 (2005): 141–54.

—, "Russell's Corrected Page Proofs of *Principia Mathematica*". *Russell* 39 (2019): 141–66.

LOVE, A. E. H. *Theoretical Mechanics: an Introductory Treatise on the Principles of Dynamics, with Applications and Numerous Examples*. Cambridge: Cambridge U. P., 1897.

MACCOLL, HUGH. "Symbolical Reasoning". *Mind* 5 (1880): 45–60.

PEANO, GIUSEPPE. *Formulaire de mathématiques* [i.e., *Formulario mathematico*], Vol. 1. Turin: Bocca Frères, 1895.

RUSSELL, BERTRAND. "Sur la logique des relations avec des applications à la théorie des séries". *Revue de mathématiques*, Turin, 7, nos. 2–3 (1901): 115–36, 137–48; [as "The Logic of Relations with Some Applications to the Theory of Series"] **8** in *Papers* 3.

—. "On Likeness". Ms. (1902); **15** in *Papers* 3.

—. "Classes". Ms. (1903); **1** in *Papers* 4.

—. "The Theory of Implication". *American Journal of Mathematics* 28 (April 1906): 159–202; **1** in *Papers* 5.

—. "Mathematical Logic as Based on the Theory of Types". *American Journal of Mathematics* 30 (May 1908): 222–62; **22** in *Papers* 5.

—. *EFG*. *Papers* 2, 3, 4, 5.

—, AND A. N. WHITEHEAD. *PM*.

SCHLIMM, DIRK. "Peano on Symbolization, Design Principles for Notations, and the Dot Notation". *Philosophia Scientiae* 25, no. 1 (2021): 95–126.

SCHRÖDER, ERNST. *Volesungen über die Algebra der Logik*. Vol. 2. Leipzig: B. G. Teubner, 1891.

SCOTT, CHARLOTTE ANGAS. *An Introductory Account of Certain Modern Ideas and Methods in Plane Analytical Geometry*. London and New York: Macmillan, 1894.

TODHUNTER, ISAAC. *Spherical Trigonometry, for the Use of Colleges and Schools*. 3rd ed. London and New York: Macmillan, 1871.

TURING, A. M. "The Use of Dots as Brackets in Church's System". *The Journal of Symbolic Logic* 7 (1942): 146–56.

WHITTAKER, E. T. *A Course of Modern Analysis: an Introduction to the General Theory of Infinite Series and of Analytic Functions; with an Account of the Principal Transcendental Functions*. Cambridge: Cambridge U. P., 1902.